

Computation of Lie Transformations from a Power Series: Bounds and Optimum Truncation

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Abstract

The problem considered is the computation of an infinite product (composition) of Lie transformations generated by homogeneous polynomials of increasing order from a given convergent power series. Bounds are computed for the infinitesimal form of Lie transformations. The results obtained do not guarantee convergence of the product. Instead, the optimum truncation is determined by minimizing the terms of order $n + 1$ that remain after the first n Lie transformations have been applied.

I. INTRODUCTION

A method based on an infinite product (composition) of Lie transformations (exponentiated vector fields) generated by homogeneous polynomials of increasing order was developed a long time ago in order to efficiently perform perturbative calculations in Hamiltonian systems when the small parameters are the dynamical variables themselves [1]. In particular, it was shown in Ref. [1] that the product can be computed from, or be used to compute, a power series in the dynamical variables. The relation between Lie transformations and power series, however, was established only at a formal level, that is order by order.

More recent work has provided firm bounds on the results that can be obtained using the method. In Refs. [2,3,4], for example, a variant of the method is applied to the problem of bringing a Hamiltonian function or a Hamiltonian vector field to normal form. In Ref. [5], and in Ref. [4] for Hamiltonian systems, sufficient conditions are given on the coefficients of the polynomials and on the domain of the dynamical variables such that the infinite product of Lie transformations is convergent. In this paper we turn to the construction of Lie transformations from a power series. Assuming that the power series has a finite domain of absolute convergence, we obtain bounds for the norms of the vector fields that are computed from such a series and develop a procedure for determining the optimum truncation of the product of Lie transformations. (Note that the analogous problem with power series for Refs. [2] and [3] is the one in which only the first nonlinear term in the series is nonvanishing.) As in Ref. [5], we do not require that Lie transformations be symplectic (i.e. that they arise from Hamiltonian systems); rather, the vector fields are taken to be arbitrary homogeneous polynomials.

In Section II we introduce notation, which for quantities that appear in both is the same as in Ref. [5], and write down an expression for the coefficients of the polynomials in terms of the coefficients of the power series. Section III contains two lemmas which allow us to pass from the expression for the coefficients to an inequality in the form of a recursion relation for the norms of the vector fields. In Section IV we then use the recursion relation to obtain

a bound on the norms, which is the main result of the paper. We also provide there an asymptotic expression for the bound valid in the limit $n \rightarrow \infty$, where n is the order of the polynomial. The question of optimum truncation of the product is considered in Section V, and a summary of the results is given in Section VI.

II. PRELIMINARIES

We work with the transformation \mathcal{M} formally defined by

$$\mathcal{M}z = e^{L_2(z)} e^{L_3(z)} \dots e^{L_n(z)} \dots z, \quad z \in \mathbb{C}^d. \quad (2.1)$$

Here L_n is a vector field

$$L_n(z) = \sum_{j=1}^d g_j^{(n)}(z) \frac{\partial}{\partial z_j}, \quad (2.2)$$

and $g_j^{(n)}$ a homogeneous polynomial in z of order n ,

$$g_j^{(n)}(z) = \sum_{|r|=n} a_{rj}^{(n)} z^r. \quad (2.3)$$

The subscript r stands for the collection of indices r_1, \dots, r_d , $|r| \stackrel{\text{def}}{=} r_1 + \dots + r_d$, and $z^r \stackrel{\text{def}}{=} z_1^{r_1} \dots z_d^{r_d}$. The exponential of L_n is given by the usual infinite series

$$e^{L_n(z)} = \sum_{s=0}^{\infty} \frac{1}{s!} [L_n(z)]^s, \quad (2.4)$$

where $s = 0$ corresponds to the identity transformation. In the definition of \mathcal{M} the linear transformation has been set equal to the identity, as its computation is not germane to the problem at hand. The designation of Eq. (2.1) as formal, on the other hand, reflects the fact that we have not specified a domain in \mathbb{C}^d , if such one exists, on which the infinite series of Eq. (2.4) are convergent for $n = 2, 3, \dots$. We also define \mathcal{M}_n as the product of Lie transformations of the form (2.1) truncated at order n .

The properties of Lie transformations and their use in perturbation calculations are not discussed further in this paper. The interested reader is instead directed to Refs. [6] for a sampling of the surveys of the subject.

Suppose we are given the power series

$$P_k(z) = z_k + \sum_{i=2}^{\infty} \sum_{|r|=i} b_{rk}^{(i)} z^r \quad (2.5)$$

which has a nonvanishing domain of absolute convergence, denoted here by \mathcal{D} . (\mathcal{D} evidently includes the origin.) We are going to examine the construction of vector fields L_n chosen in such a way that $\mathcal{M}z$ and $P(z)$ agree order by order in z . For the moment we do not specify the domain over which the agreement occurs. A lower bound on this domain as a function of n is given in Section V. In parallel with setting the linear transformation in \mathcal{M} equal to the identity, we have assumed that to first order in z $P(z) = z$.

For a vector v , regardless of the vector space, we define the norm $\|v\|$ by

$$\|v\| = \max_i |v_i|, \quad (2.6)$$

where $|\cdot|$ stands for the modulus. For brevity we denote the norm of z by x , $x = \|z\|$. We also define the quantity $\alpha_j^{(n)}$ by

$$\alpha_j^{(n)} = \sum_{|r|=n} |a_{rj}^{(n)}|, \quad (2.7)$$

and α_n by $\alpha_n = \|\alpha^{(n)}\|$. The following relation holds:

$$\|L_n\| = \max_j |g_j^{(n)}(z)| = \max_j \left| \sum_{|r|=n} a_{rj}^{(n)} z^r \right| \leq \alpha_n x^n \quad (2.8)$$

In the subsequent sections we will obtain a bound for α_n which will thus enable us to place a bound on $\|L_n\|$.

The first step is to write down an expression for the coefficients $a_{rj}^{(n)}$ in terms of the coefficients $b_{rk}^{(n)}$. We expand the Lie transformations into power series and match terms of the same order in z to get

$$\sum_{[n]=n-1} \frac{L_2^{s_2} \dots L_n^{s_n}}{s_2! \dots s_n!} z_k = \sum_{|r|=n} b_{rk}^{(n)} z^r. \quad (2.9)$$

The symbol $\sum_{[p]=q}$ is defined as a sum over s_2, \dots, s_p with a condition,

$$\sum_{[p]=q} = \sum_{\substack{s_2, \dots, s_p \geq 0 \\ s_2 + 2s_3 + \dots + (p-1)s_p = q}}. \quad (2.10)$$

Note that the operators $L_i^{s_i}$ and $L_j^{s_j}$ $i \neq j$ do not commute, so the ordering is important.

In Eq. (2.9) s_n can take on only the values of 0 and 1. Together with the fact that $L_n z_k = g_k^{(n)}(z)$, this allows us to transform Eq. (2.9) into a recursion relation for $a_{rk}^{(n)}$,

$$a_{rk}^{(2)} = b_{rk}^{(2)}, \quad (2.11a)$$

$$a_{rk}^{(n)} = b_{rk}^{(n)} - \frac{\partial_z^r}{r!} \sum_{[n-1]=n-1} \frac{L_2^{s_2} \dots L_{n-1}^{s_{n-1}}}{s_2! \dots s_{n-1}!} z_k; \quad n \geq 3, \quad n \in \mathbb{N}, \quad (2.11b)$$

where $\frac{\partial_z^r}{r!} \stackrel{\text{def}}{=} \frac{\partial_{z_1}^{r_1} \dots \partial_{z_d}^{r_d}}{r_1! \dots r_d!}$. (Throughout the paper \mathbb{N} is taken to include 0.) This is the starting point for the computation of estimates for α_n .

III. A RECURSION RELATION FOR NORMS

With the definition

$$\beta_n = \max_j \left(\sum_{|r|=n} |b_{rj}^{(n)}| \right), \quad (3.1)$$

Eq. (2.11b) yields

$$\begin{aligned} \alpha_n &= \max_k \sum_{|r|=n} \left| b_{rk}^{(n)} - \frac{\partial_z^r}{r!} \sum_{[n-1]=n-1} \frac{L_2^{s_2} \dots L_{n-1}^{s_{n-1}}}{s_2! \dots s_{n-1}!} z_k \right| \\ &\leq \beta_n + \max_k \sum_{|r|=n} \left| \sum_{[n-1]=n-1} \frac{\partial_z^r}{r!} \frac{L_2^{s_2} \dots L_{n-1}^{s_{n-1}}}{s_2! \dots s_{n-1}!} \right| \\ &\leq \beta_n + \sum_{[n-1]=n-1} \frac{1}{s_2! \dots s_{n-1}!} \max_k \sum_{|r|=n} \left| \frac{\partial_z^r}{r!} L_2^{s_2} \dots L_{n-1}^{s_{n-1}} z_k \right| \end{aligned} \quad (3.2)$$

Note that the component of the vector on the right side is determined by the component of z and is labeled here by k . Our goal is to express the right side of the last inequality in (3.2) in terms of $\alpha_2, \dots, \alpha_{n-1}$. We accomplish this through two lemmas (the second one will also be used in Section V).

Consider a vector function F whose components are homogeneous polynomials,

$$F_k^{(l)} = \sum_{|i|=l} f_{ik}^{(l)} z^i; \quad 1 \leq k \leq d, \quad (3.3)$$

and for which

$$\max_k \left(\sum_{|i|=l} |f_{ik}^{(l)}| \right) \leq \phi_l \quad (3.4)$$

for some $\phi_l \in \mathbb{R}^+$. Define the quantities c , m , and B by

$$\sum_{|t|=m(\phi_l x^l, s_2, \dots, s_n)} c_{tk}(F, s_2, \dots, s_n) z^t = L_2^{s_2} \dots L_n^{s_n} F_k^{(n)}(z), \quad (3.5a)$$

$$B(\phi_l x^l, s_2, \dots, s_n) x^{m(\phi_l x^l, s_2, \dots, s_n)} = (x^2 \alpha_2 \frac{d}{dx})^{s_2} \dots (x^n \alpha_n \frac{d}{dx})^{s_n} \phi_l x^l. \quad (3.5b)$$

Here use is made of the obvious fact that the power of x in (3.5b) is the same as the power of z in (3.5a). Evidently $B(\phi_l x^l, s_2, \dots, s_n)$ is a nonnegative real quantity and $m(\phi_l x^l, s_2, \dots, s_n)$ is a nonnegative integer. The arguments of c , m , and B have been chosen to be rather explicit, so that the notation is sufficiently general for the manipulations that follow. When referring to a power of only one vector field, on the other hand, we drop the subscript on the summation index, denoting it by s , and replace the argument $\phi_l x^l$ of m and B simply by l . The following holds:

Lemma 3.1. For all $s \in \mathbb{N}$ and $l \in \mathbb{N}$

$$\max_k \left(\sum_{|t|=m(l,s)} |c_{tk}(F, s)| \right) \leq B(l, s). \quad (3.6)$$

and

$$\|L_n^s F^{(l)}(z)\| \leq B(l, s) x^{m(l,s)} \quad (3.7)$$

This lemma is given in Refs. [7], though its proof is only outlined there. In the Appendix we provide a more complete proof (which is an extension of the relations derived in Appendix A of Ref. [5]).

The product of operators of the form $L_i^{s_i}$ can now be bounded by the lemma below.

Lemma 3.2. For all $n \geq 2$, $n \in \mathbb{N}$ and all functions F of the form (3.3), $l \in \mathbb{N}$,

$$\max_k \sum_{|t|=m(\phi_l x^l, s_2, \dots, s_n)} |c_{tk}(F, s_2, \dots, s_n)| \leq B(\phi_l x^l, s_2, \dots, s_n) \quad (3.8)$$

and

$$\|L_2^{s_2} \dots L_n^{s_n} F^{(l)}(z)\| \leq B(\phi_l x^l, s_2, \dots, s_n) x^{m(\phi_l x^l, s_2, \dots, s_n)}. \quad (3.9)$$

Proof is by induction on n and is straightforward. For $n = 2$ inequalities (3.8,3.9) are the same as inequalities (3.6,3.7) and thus hold by Lemma 3.1. Assume now (3.8,3.9) hold for a fixed n and all functions F of the form (3.3). Then

$$\|L_2^{s_2} \dots L_n^{s_n} L_{n+1}^{s_{n+1}} F^{(l)}(z)\| = \|L_2^{s_2} \dots L_n^{s_n} \tilde{F}^{(l')}(z)\| \quad (3.10)$$

where

$$\tilde{F}_k^{(l')}(z) = \sum_{|t|=l'} c_{tk}(F, s_{n+1}) z^t \quad (3.11)$$

with $l' = ns_{n+1} + l$ and, by Lemma 3.1,

$$\max_k \sum_{|t|=l'} |c_{tk}(F, s_{n+1})| \leq B(\phi_l x^l, s_{n+1}). \quad (3.12)$$

Use of the induction assumption yields

$$\max_k \sum_{|t|=m(B(\phi_l x^l, s_{n+1})x^{l'}, s_2, \dots, s_n)} |c_{tk}(\tilde{F}, s_2, \dots, s_n)| \leq B(B(\phi_l x^l, s_{n+1})x^{l'}, s_2, \dots, s_n) \quad (3.13)$$

and

$$\|L_2^{s_2} \dots L_n^{s_n} \tilde{F}^{(l')}(z)\| \leq B(B(\phi_l x^l, s_{n+1})x^{l'}, s_2, \dots, s_n) x^{m(B(\phi_l x^l, s_{n+1})x^{l'}, s_2, \dots, s_n)}. \quad (3.14)$$

By unfolding the definitions of m and B we get

$$\begin{aligned} & B(B(\phi_l x^l, s_{n+1})x^{l'}, s_2, \dots, s_n) x^{m(B(\phi_l x^l, s_{n+1})x^{l'}, s_2, \dots, s_n)} \\ &= (x^2 \alpha_2 \frac{d}{dx})^{s_2} \dots (x^n \alpha_n \frac{d}{dx})^{s_n} B(\phi_l x^l, s_{n+1}) x^{l'} \\ &= (x^2 \alpha_2 \frac{d}{dx})^{s_2} \dots (x^n \alpha_n \frac{d}{dx})^{s_n} (x^{n+1} \alpha_{n+1} \frac{d}{dx})^{s_{n+1}} \phi_l x^l \\ &= B(\phi_l x^l, s_2, \dots, s_n, s_{n+1}) x^{m(\phi_l x^l, s_2, \dots, s_n, s_{n+1})}, \end{aligned} \quad (3.15)$$

and so replace $B(B(\phi_l x^l, s_{n+1})x', s_2, \dots, s_n)$ by $B(\phi_l x^l, s_2, \dots, s_n, s_{n+1})$ and $m(B(\phi_l x^l, s_{n+1})x', s_2, \dots, s_n)$ by $m(\phi_l x^l, s_2, \dots, s_n, s_{n+1})$ in inequalities (3.13, 3.14). With the further replacement of $c_{tk}(\tilde{F}, s_2, \dots, s_n)$ by $c_{tk}(F, s_2, \dots, s_n, s_{n+1})$, the proof is complete.

□

We can now make progress with inequality (3.2). Since

$$\max_k \sum_{|r|=n} \left| \frac{\partial^r}{\partial z^r} L_2^{s_2} \dots L_{n-1}^{s_{n-1}} z_k \right| = \max_k \sum_{|r|=n} |c_{rk}(z, s_2, \dots, s_{n-1})|, \quad (3.16)$$

use of inequality (3.8) yields

$$\alpha_n \leq \beta_n + \sum_{[n-1]=n-1} \frac{1}{s_2! \dots s_{n-1}!} B(x, s_2, \dots, s_{n-1}). \quad (3.17)$$

Note that the apparent dependence of the right side of (3.2) on z (or $\|z\|$) has disappeared, as it should.

The final step is to obtain an explicit expression for $B(x, s_2, \dots, s_{n-1})$, which requires the evaluation of the right side of Eq. (3.5b) for $l = 1$, $\phi_l = 1$. First we note that

$$\left(x^n \frac{d}{dx}\right)^s x^p = \frac{(n-1)^s \Gamma\left(\frac{s(n-1)+p}{n-1}\right)}{\Gamma\left(\frac{p}{n-1}\right)} x^{p+s(n-1)}, \quad (3.18)$$

where we take, as it is sufficient for our purposes, n, s , and p to be integers, with $n \geq 2$, $s \geq 0$, and $p \geq 1$. The relation (3.18) is easily proven by induction on s . Repeated use of (3.18) on the right side of Eq. (3.5b) then leads to

$$\begin{aligned} B(x, s_2, \dots, s_{n-1}) &= \alpha_2^{s_2} (2\alpha_3)^{s_3} \dots ((n-2)\alpha_{n-1})^{s_{n-1}} \\ &\times \frac{\Gamma\left(\frac{1+s_{n-1}(n-2)}{n-2}\right) \Gamma\left(\frac{1+s_{n-1}(n-2)+s_{n-2}(n-3)}{n-3}\right)}{\Gamma\left(\frac{1}{n-2}\right) \Gamma\left(\frac{1+s_{n-1}(n-2)}{n-3}\right)} \dots \frac{\Gamma\left(\frac{1+s_{n-1}(n-2)+\dots+s_2}{1}\right)}{\Gamma\left(\frac{1+s_{n-1}(n-2)+\dots+2s_3}{1}\right)}, \end{aligned} \quad (3.19)$$

which is the desired, though admittedly cumbersome, expression for B .

With the definitions $\eta_n = n\alpha_{n+1}$, $\tau_n = n\beta_{n+1}$,

$$\begin{aligned} Q_m &= 1; & m &= n-1 \\ Q_m &= 1 + \sum_{j=1}^{n-m-1} (n-j)s_{n-j+1}; & 1 \leq m &\leq n-2, \end{aligned} \quad (3.20a)$$

and

$$\mathcal{G}(s_2, \dots, s_n) = \prod_{m=1}^{n-1} \frac{\Gamma(s_{m+1} + \frac{Q_m}{m})}{s_{m+1}! \Gamma(\frac{Q_m}{m})}, \quad (3.20b)$$

inequality (3.17) and Eq. (2.11a) become

$$\eta_1 = \tau_1 \quad (3.21a)$$

$$\eta_n \leq \tau_n + n \sum_{[n]=n} \eta_1^{s_2} \dots \eta_{n-1}^{s_n} \mathcal{G}(s_2, \dots, s_n); \quad n \geq 2, \quad n \in \mathbb{N}. \quad (3.21b)$$

In the next section we will use these relations to get a bound for η_n . We call attention to the interesting fact that the relations (3.21), and hence the results that follow, do not depend explicitly on d . The dimensionality of the space enters only through the definition of quantities η_n and τ_n .

IV. BOUND FOR η_N

Inequality (3.21b) is a complicated relation between η 's and τ 's. The reader is invited to show that attempts to establish simple estimates for η_n , such as $\eta_n \leq K^n$ or $\eta_n \leq K^n n!$, $K \in \mathbb{R}^+$, by induction from (3.21b) fail. (For the latter case note that the sums over s always contain the term $s_2 = s_n = 1$, $s_i = 0$; $3 \leq i \leq n-1$, for which $\mathcal{G}(1, 0, \dots, 0, 1) = \frac{n}{n-1}$.) The following gives a bound for η_n .

Theorem 4.1. Let $K = \max_n \tau_n^{\frac{1}{n}}$ and define the quantities h_n by

$$h_1 = 1; \quad h_n = \prod_{j=2}^n (1 + 2^{\frac{j}{j-1}} (j-1))^{\frac{1}{j}}; \quad n \geq 2, \quad n \in \mathbb{N}. \quad (4.1)$$

Then

$$\eta_n \leq K^n h_n^n. \quad (4.2)$$

PROOF. First, since $P(z)$ of Eq. (2.5) is a convergent power series, $|b_{rk}^{(n)}|$ is bounded by an exponentially growing function of n . As

$$\beta_n \leq \binom{n+d-1}{n} \max_{rk} |b_{rk}^{(n)}|, \quad (4.3)$$

β_n is also bounded by an exponentially growing function of n , and so is τ_n . Thus the quantity $K = \max_n \tau_n^{\frac{1}{n}}$ for $n \geq 2$, $n \in \mathbb{N}$, exists and is finite.

Next, we manipulate the ratios of Γ functions that appear in \mathcal{G} . For $1 \leq m \leq n-1$,

$$\begin{aligned} \frac{\Gamma(s_{m+1} + \frac{Q_m}{m})}{\Gamma(\frac{Q_m}{m})} &= (s_{m+1} - 1 + \frac{Q_m}{m})(s_{m+1} - 2 + \frac{Q_m}{m}) \cdots \frac{Q_m}{m} \\ &\leq (s_{m+1} - 1 + \frac{Q_m}{m})^{s_{m+1}} \\ &= \frac{1}{m^{s_{m+1}}} [m(s_{m+1} - 1) + Q_m]^{s_{m+1}} \\ &\leq \frac{1}{m^{s_{m+1}}} (n + 1 - m)^{s_{m+1}}. \end{aligned} \quad (4.4)$$

The last inequality makes use of $ms_{m+1} + Q_m \leq n + 1$, which follows from the condition on the sums over s . Hence,

$$\mathcal{G}(s_2, \dots, s_n) \leq \frac{n^{s_2} (n-1)^{s_3} \cdots 2^{s_n}}{s_2! \cdots s_n! 1^{s_2} 2^{s_3} \cdots (n-1)^{s_n}} \quad (4.5)$$

and inequality (3.21b) becomes

$$\eta_n \leq \tau_n + n \sum_{[n]=n} \frac{(n\eta_1)^{s_2} [(n-1)\eta_2]^{s_3} \cdots (2\eta_{n-1})^{s_n}}{s_2! \cdots s_n! 1^{s_2} \cdots (n-1)^{s_n}}. \quad (4.6)$$

To proceed further we establish the following statement.

Lemma 4.1. For $n \geq 2$ and $1 \leq m \leq n-1$

$$(n-m+1)^{\frac{1}{m}} h_m \leq 2^{\frac{1}{n-1}} h_{n-1}. \quad (4.7)$$

Proof of Lemma 4.1 is effected in five steps.

(i) For the case $m = n-1$ (4.7) obviously holds. Since $m = n-1$ is the only value of m when $n = 2$, it remains to consider $n \geq 3$, $1 \leq m \leq n-2$.

For further manipulations it is useful to denote the ratio of the left and right sides of (4.7) by G ,

$$G(n, m) = \frac{(n-m+1)^{\frac{1}{m}} h_m}{2^{\frac{1}{n-1}} h_{n-1}} = \frac{(n-m+1)^{\frac{1}{m}}}{2^{\frac{1}{n-1}} \prod_{j=m+1}^{n-1} (1 + 2^{\frac{j}{j-1}} (j-1))^{\frac{1}{j}}}. \quad (4.8)$$

We thus need to show that $G(n, m) \leq 1$.

(ii) Let $m = n - 2$. For $n = 3$ the direct calculation shows that $G(3, 1) = 0.95$, whereas for $n \geq 4$ we have

$$\begin{aligned} G(n, n-2) &= \frac{3^{\frac{1}{n-2}}}{2^{\frac{1}{n-1}}[1 + 2^{\frac{n-1}{n-2}}(n-2)]^{\frac{1}{n-1}}} < \frac{3^{\frac{1}{n-2}}}{2^{\frac{1}{n-1}}(1 + 2^{\frac{n-1}{n-2}})^{\frac{1}{n-1}}} \\ &< \left(\frac{3}{2^{\frac{2n-3}{n-1}}}\right)^{\frac{1}{n-2}} \leq \left(\frac{3}{2^{\frac{5}{3}}}\right)^{\frac{1}{n-2}} = 0.94^{\frac{1}{n-2}}. \end{aligned} \quad (4.9)$$

The last inequality uses $\frac{2n-3}{n-1} \geq \frac{5}{3}$ which evidently holds for $n \geq 4$. It remains to consider $1 \leq m \leq n - 3$; $n \geq 4$.

(iii) Let $m = 1$. By direct calculation $G(4, 1) = 0.75$, whereas for $n \geq 5$ we manipulate the product appearing in the definition of G to get

$$\begin{aligned} \prod_{j=2}^{n-1} (1 + 2^{\frac{j}{j-1}}(j-1))^{\frac{1}{j}} &= \exp\left[\sum_{j=2}^{n-1} \frac{1}{j} \log(1 + 2^{\frac{j}{j-1}}(j-1))\right] > \exp\left[\sum_{j=2}^{n-1} \frac{1}{j} \log(2j-1)\right] \\ &\geq \exp\left[\frac{n-2}{n-1} \log(2n-3)\right] = (2n-3)^{\frac{n-2}{n-1}}. \end{aligned} \quad (4.10)$$

The last inequality follows from the monotonic decrease of the summand as a function of j . We provide here a brief justification of this argument which, with slight modifications, also appears in step (v). Treating j as a continuous variable, we have

$$\frac{d\left[\frac{1}{j} \log(2j-1)\right]}{dj} = \frac{(2j-1)(1 - \log(2j-1)) + 1}{j^2(2j-1)}. \quad (4.11)$$

For $j = 2$ the value of the numerator is -0.70 , and this value obviously decreases with increasing j .

Returning to the bounding of G , since $n \geq 5$ we write $2n-3 \geq \frac{7}{5}n$ and use inequality (4.10) to get

$$G(n, 1) < \left[\frac{1}{\left(\frac{7}{5}\right)^{n-2} \frac{2}{n}}\right]^{\frac{1}{n-1}}. \quad (4.12)$$

Note that if n is treated as a continuous variable then

$$\frac{d\left[\left(\frac{7}{5}\right)^{n-2} \frac{2}{n}\right]}{dn} = \frac{50\left(\frac{7}{5}\right)^n}{49n} \left[\log\left(\frac{7}{5}\right) - \frac{1}{n}\right], \quad (4.13)$$

and so for $n \geq \frac{1}{\log(\frac{7}{5})} = 2.97$, the denominator is a monotonically increasing function of n . At $n = 5$ it takes the value 1.10. Therefore, $G(n, 1) < 1$ for $n \geq 5$. It remains to consider $2 \leq m \leq n - 3$; $n \geq 5$.

(iv) By direct calculation $G(5, 2) = 0.52$, and we are left to explore only $2 \leq m \leq n - 3$; $n \geq 6$.

(v) Now we can consider the remaining values of m and n . Proceeding in analogy with Eq. (4.10), we write

$$\prod_{j=m+1}^{n-1} (1 + 2^{\frac{j}{j-1}}(j-1))^{\frac{1}{j}} > (2n-3)^{\frac{n-m-1}{n-1}} > n^{\frac{n-m-1}{n-1}}. \quad (4.14)$$

Therefore,

$$G(n, m) < \frac{(n-m+1)^{\frac{1}{m}}}{2^{\frac{1}{n-1}} n^{\frac{n-m-1}{n-1}}} < \frac{1}{2^{\frac{1}{n-1}}} n^{\frac{1}{m(n-1)}(n-1-m(n-1)+m^2)}. \quad (4.15)$$

We now examine the parabola

$$\mathcal{P}(m) = m^2 - m(n-1) + n-1, \quad (4.16)$$

where m takes on all real values. Zeroes of $\mathcal{P}(m)$ are located at

$$m_{+/-} = \frac{1}{2}[n-1 \pm \sqrt{(n-1)^2 - 4n+4}]. \quad (4.17)$$

The quantity under the square root satisfies

$$n^2 - 6n + 5 = (n-4)^2 + 2n - 11 > (n-4)^2 \quad \text{for } n \geq 6, \quad (4.18)$$

and so $m_- < \frac{3}{2}$ and $m_+ > n - \frac{5}{2}$. Therefore for $2 \leq m \leq n - 3$, $\mathcal{P}(m) < 0$, which gives $G(n, m) < 1$.

Putting together the results of (i)–(v) completes the proof of Lemma 4.1. \square

To finish the proof of Theorem 4.1 we carry out an induction on n . For $n = 1$ we have

$$\eta_1 = \tau_1 \leq K, \quad (4.19)$$

which verifies (4.2). Assume now that (4.2) holds through $n - 1$. Then inequality (4.6) gives

$$\begin{aligned}
\eta_n &\leq \tau_n + nK^n \sum_{[n]=n} n^{s_2} [(n-1)^{\frac{1}{2}} h_2]^{2s_3} \dots [2^{\frac{1}{n-1}} h_{n-1}]^{(n-1)s_n} \frac{1}{s_2! \dots s_n! 1^{s_2} \dots (n-1)^{s_n}} \\
&\leq \tau_n + nK^n \sum_{[n]=n} [2^{\frac{1}{n-1}} h_{n-1}]^{s_2+2s_3+\dots+(n-1)s_n} \frac{1}{s_2! \dots s_n! 1^{s_2} \dots (n-1)^{s_n}} \\
&\leq \tau_n + nK^n 2^{\frac{n}{n-1}} h_{n-1}^n \sum_{[n]=n} \frac{1}{s_2! \dots s_n! 1^{s_2} \dots (n-1)^{s_n}}, \tag{4.20}
\end{aligned}$$

where we have used the condition on the sum to sum the power of the summand. The second inequality follows from Lemma 4.1. The remaining sums over s nicely sum to $1 - \frac{1}{n}$, as can be demonstrated using Cauchy's identity [8]

$$\sum_{[n+1]=n} \frac{1}{s_2! \dots s_n! s_{n+1}! 1^{s_2} \dots (n-1)^{s_n} n^{s_{n+1}}} = 1. \tag{4.21}$$

Substitution of this result into Eq. (4.20) leads to the inequalities

$$\begin{aligned}
\eta_n &\leq \tau_n + K^n 2^{\frac{n}{n-1}} h_{n-1}^n (n-1) \leq K^n \left[1 + 2^{\frac{n}{n-1}} h_{n-1}^n (n-1) \right] \\
&\leq K^n \left[h_{n-1} (1 + 2^{\frac{n}{n-1}} (n-1))^{\frac{1}{n}} \right]^n = K^n h_n^n. \tag{4.22}
\end{aligned}$$

For the second inequality we have used $K^n \geq \tau_n$ and for the third one $h_{n-1} \geq 1$. This completes the proof of Theorem 4.1. \square

The large- n behavior of the estimate (4.2) is not easy to discern. We thus provide an asymptotic expression for the result. First we convert the product appearing in (4.1) into a sum by taking the logarithm of h_n^n , and then use the Euler-Maclaurin Sum Formula [9] to get

$$\begin{aligned}
\log h_n &= \sum_{k=0}^m \theta(k) \\
&\sim \frac{1}{2} \theta(m) + \int_0^m \theta(t) dt + c_1 + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{B_{j+1}}{(j+1)!} \frac{d^j \theta(m)}{dm^j}; \quad m \rightarrow \infty, \tag{4.23a}
\end{aligned}$$

where

$$\theta(k) = \frac{1}{k+2} \log(1 + 2^{\frac{k+2}{k+1}} (k+1)), \tag{4.23b}$$

B_n are the Bernoulli numbers, $m = n-2$ and c_1 is a constant to be determined. The integral of θ evaluates to

$$\int_0^m \theta(t) dt = \frac{1}{2} [\log(m+2)]^2 + \log 2 \log(m+2) + c_2 + \frac{1}{m+2} \left(\frac{1}{2} - \log 2 \right) + O\left(\frac{1}{m^2}\right), \quad (4.24)$$

where c_2 is given by $c_2 = -0.577765\dots$. The terms in the sum over j , on the other hand, are of order $\frac{\log(n)}{n^2}$ and can be neglected. Nevertheless, for the numerical computation of the constant c_1 we have used the $j = 1$ term in order to improve the numerical convergence of the procedure (the $j = 2$ term is zero since $B_3 = 0$). The resulting asymptotic expansion for the bound on η_n reads

$$K^n h_n^n \sim (K c_3)^n \sqrt{\frac{e n}{2}} n^{n(\log 2 + \frac{1}{2} \log n)} \times \exp \left[O\left(\frac{\log n}{n}\right) \right]; \quad n \rightarrow \infty. \quad (4.25)$$

The constant c_3 is given by $c_3 = \exp(c_1 + c_2)$ and takes the value $c_3 = 0.857695\dots$

V. OPTIMUM TRUNCATION

In Ref. [5] we have shown that sufficient conditions for convergence of $\mathcal{M}_n z$ as $n \rightarrow \infty$ are that η_n be bounded by an exponentially growing function of n and that z be restricted to a suitable domain around the origin. Yet, inequality (3.21b) is not consistent with $\eta_n \leq K^n$, $K \in \mathbb{R}^+$. Instead of considering convergence, then, we turn to the asymptotic properties of $\mathcal{M}_n z$. In particular, we examine the question of optimum truncation.

The most natural quantity to optimize is the difference between $P(z)$ and $\mathcal{M}_n z$. Denoting this quantity by $R(n)$, we have

$$\begin{aligned} R(n) &= \|P(z) - e^{L_2} \dots e^{L_n} z\| \\ &= \left\| \sum_{q=n+1}^{\infty} \sum_{|r|=q} b_{rk}^{(q)} z^r - \sum_{q=n+1}^{\infty} \sum_{[n]=q-1} \frac{L_2^{s_2} \dots L_n^{s_n}}{s_2! \dots s_n!} z_k \right\| \\ &\leq \sum_{q=n+1}^{\infty} x^q \left[\beta_q + \sum_{[n]=q-1} \frac{B(x, s_2, \dots, s_n)}{s_2! \dots s_n!} \right] \\ &= \sum_{q=n+1}^{\infty} x^q \left[\beta_q + \sum_{[n]=q-1} \eta_1^{s_2} \dots \eta_{n-1}^{s_n} \mathcal{G}(s_2, \dots, s_n) \right]. \end{aligned} \quad (5.1)$$

For the inequality we have used Lemma 3.2 and the condition on the sums over s , and for the last equality Eq. (3.19). Note that $R(n)$ is not necessarily defined on the entire domain \mathcal{D} . This question will be addressed later in this section. We could now substitute

the result of Theorem 4.1 in the last expression in Eq. (5.1) to get an explicit estimate for $R(n)$. The resulting expression, however, involves an infinite sum (where each term is very complicated). We thus use additional inequalities to obtain a closed-form estimate for $R(n)$.

The first to be simplified is the ratio of Γ functions occurring in \mathcal{G} . (It is not fruitful to use Eq. (4.4) here because the condition on the sums over s is different.) For $1 \leq m \leq n-1$ the following holds:

$$\begin{aligned}
\frac{\Gamma(s_{m+1} + \frac{Q_m}{m})}{\Gamma(\frac{Q_m}{m})} &= (s_{m+1} - 1 + \frac{Q_m}{m})(s_{m+1} - 2 + \frac{Q_m}{m}) \cdots \frac{Q_m}{m} \\
&= \frac{1}{m^{s_{m+1}}} (ms_{m+1} - m + Q_m)(ms_{m+1} - 2m + Q_m) \cdots Q_m \\
&\leq \frac{1}{m^{s_{m+1}}} (q - m)(q - 2m) \cdots (q - s_{m+1}m) \\
&\leq \frac{1}{m^{s_{m+1}}} (q - 1)(q - 2) \cdots (q - s_{m+1}) \\
&= \frac{1}{m^{s_{m+1}}} \frac{(q - 1)!}{(q - 1 - s_{m+1})!}.
\end{aligned} \tag{5.2}$$

The first inequality relies on the relation $ms_{m+1} + Q_m \leq q$. Using (5.2) and (4.2), the sums over s in Eq. (5.1) become

$$\begin{aligned}
&\sum_{[n]=q-1} \eta_1^{s_2} \cdots \eta_{n-1}^{s_n} \mathcal{G}(s_2, \dots, s_n) \\
&\leq (K2^{\frac{1}{n-1}} h_{n-1})^{q-1} \sum_{[n]=q-1} \frac{\mathcal{G}(s_2, \dots, s_n)}{n^{s_2} (n-1)^{s_3} \cdots 2^{s_n}} \\
&\leq (K2^{\frac{1}{n-1}} h_{n-1})^{q-1} \sum_{[n]=q-1} \binom{q-1}{s_2} \frac{1}{n^{s_2}} \binom{q-1}{s_3} \frac{1}{[2(n-1)]^{s_3}} \cdots \binom{q-1}{s_n} \frac{1}{[2(n-1)]^{s_n}} \\
&\leq (K2^{\frac{1}{n-1}} h_{n-1})^{q-1} \sum_{s_2=0}^{q-1} \cdots \sum_{s_n=0}^{q-1} \binom{q-1}{s_2} \frac{1}{n^{s_2}} \binom{q-1}{s_3} \frac{1}{[2(n-1)]^{s_3}} \cdots \binom{q-1}{s_n} \frac{1}{[2(n-1)]^{s_n}} \\
&\leq (K2^{\frac{1}{n-1}} h_{n-1})^{q-1} \left[\prod_{j=1}^{n-1} \left(1 + \frac{1}{j(n-j+1)} \right) \right]^{q-1} \\
&\leq (K2^{\frac{1}{n-1}} h_{n-1})^{q-1} \left(1 + \frac{1}{n} \right)^{(n-1)(q-1)} \\
&< (Ke2^{\frac{1}{n-1}} h_{n-1})^{q-1}.
\end{aligned} \tag{5.3}$$

The first inequality follows from Lemma 4.1 and the condition on the sum, the second one from Eq. (5.2), the third one is evident upon an examination of the ranges of indices s_2

through s_n subject to the condition $[n] = q - 1$, whereas the last one is clear from the relation $(1 + \frac{1}{n})^{n-1} < (1 + \frac{1}{n})^n < e$. Finally, as mentioned in the proof of Theorem 4.1, β_q is bounded by an exponentially growing function of q ,

$$\beta_q \leq \left(\frac{c}{3^{\frac{1}{3}}}\right)^{q-1}, \quad (5.4)$$

for some $c \in \mathbb{R}^+$. Consistent with Eq. (2.5) we have taken $\beta_1 = 1$. In terms of c , K satisfies $K \leq c$.

From inequality (5.3), and using Eq. (5.4), we then obtain the following closed-form estimate for $R(n)$:

$$R(n) < \frac{3^{\frac{1}{3}}}{c} \sum_{q=n+1}^{\infty} \left(\frac{cx}{3^{\frac{1}{3}}}\right)^q + \frac{1}{ce2^{\frac{1}{n-1}}h_{n-1}} \sum_{q=n+1}^{\infty} (cxe2^{\frac{1}{n-1}}h_{n-1})^q \quad (5.5a)$$

$$= \frac{x^{n+1}c^n}{3^{\frac{n}{3}}\left(1 - \frac{cx}{3^{\frac{1}{3}}}\right)} + \frac{x^{n+1}(ce2^{\frac{1}{n-1}}h_{n-1})^n}{1 - cxe2^{\frac{1}{n-1}}h_{n-1}} \stackrel{\text{def}}{=} R^*(n). \quad (5.5b)$$

Note that Eq. (5.3) or Eq. (5.5a) explicitly demonstrates the expected result that regardless of the dependence of η_n on n , one can always choose x sufficiently small to guarantee that \mathcal{M}_nz is a convergent transformation (that is that the sums of the form (2.4) acting on z converge). The condition for convergence is

$$x < \frac{1}{ce2^{\frac{1}{n-1}}h_{n-1}}, \quad (5.6)$$

which shows the shrinking of the lower bound on the domain of analyticity of \mathcal{M}_nz with increasing n . (Evidently, (5.6) is more restrictive than $z \in \mathcal{D}$. For the latter case it is sufficient that $x < 3^{\frac{1}{3}}/c$.)

Since the leading order term in $\mathcal{M}z$ is z , which is of order x , it is useful to divide R^* by x , so that the resulting quantity R^*/x can be compared to one. Note also that R^*/x depends on c and x only through the product cx , which we denote by \bar{x} .

The question of optimum truncation can now be formulated as follows: given \bar{x} , find the value of n where R^*/x reaches its minimum and find the value of the minimum. Eq. (5.5b), however, is too complicated to carry out the required calculations analytically. Instead, we

have to rely on (straightforward) numerical computations. Figure 1 shows the value of n where R^*/x reaches its minimum as a function of $-\log_{10}(\bar{x})$. This value of n is denoted by n_{\min} . Figure 2 gives the base-10 logarithm of the value of the minimum, again vs. $-\log_{10}(\bar{x})$.

If we leave rigor aside, we can obtain analytical expressions that closely approximate the solid curves in Figures 1 and 2. First we assume that the minimum of R^*/x is determined primarily by terms of lowest order in \bar{x} , \bar{x}^n and then neglect the term $(\bar{x}3^{-\frac{1}{3}})^n$ compared with $(\bar{x}e2^{\frac{1}{n-1}}h_{n-1})^n$. The latter step is justified when $(e2^{\frac{1}{n-1}}h_{n-1})^n \gg 3^{-\frac{n}{3}}$. We are thus led to examine the quantity

$$\frac{r^*(n)}{x} \stackrel{\text{def}}{=} (\bar{x}e2^{\frac{1}{n-1}}h_{n-1})^n. \quad (5.7)$$

The location and value of the minimum of r^*/x have been determined numerically and found to be in excellent agreement with the location and value of the minimum of R^*/x , the agreement improving with decreasing \bar{x} . To obtain an analytical expression for the minimum of r^*/x , however, additional approximations are needed.

We take the logarithm of $\frac{r^*}{x}$, consider n a continuous variable, differentiate with respect to it, and find the location of the minimum by setting the result equal to zero. The derivative is

$$\frac{d \log(\frac{r^*}{x})}{dn} = 1 + \log \bar{x} - \frac{\log 2}{(n-1)^2} + \log h_{n-1} + n \frac{d}{dn} \log h_{n-1}. \quad (5.8)$$

In order to evaluate the last term in (5.8) we use the identity [9]

$$\sum_{k=0}^{n-3} \theta(k) = \frac{1}{2}[\theta(0) + \theta(n-3)] + \int_0^{n-3} \theta(t)dt + \int_0^{n-3} (t - [t] - \frac{1}{2})f'(t)dt, \quad (5.9)$$

where $\theta(k)$ is given by Eq. (4.23b) and $[t]$ stands for the integer part of t , which is valid for integer n . Then we define $\log h_{n-1}$ for noninteger n to be the right side of this expression evaluated at noninteger n . Substituting (5.9) into (5.8), using the asymptotic expansion of the form (4.25) for the non-differentiated $\log h_{n-1}$, and replacing $n - [n]$ by its average value of $\frac{1}{2}$, yields

$$\frac{d \log(\frac{r^*}{x})}{dn} \sim \frac{1}{2}(\log n)^2 + (1 + \log 2) \log n + \log \bar{x} + c_4 - \frac{1}{2n} + O(\frac{\log n}{n^2}). \quad (5.10)$$

The constant c_4 is defined by $c_4 = \log c_3 + 1 + \log 2$. Both this one and the asymptotic expressions that follow are valid for large values of n . (As is evident from Eq. (5.11), and is to be expected, for the minimum of $\frac{r^*}{x}$ this is equivalent to $\bar{x} \rightarrow 0$.) For brevity we omit writing down explicitly $n \rightarrow \infty$ after each asymptotic relation.

Setting the right side of (5.10) equal to zero and neglecting terms of order $\frac{\log n}{n^2}$ gives easily

$$n' \sim \text{Int} \left\{ \exp \left[-1 - \log 2 + \sqrt{(1 + \log 2)^2 - 2 \log \bar{x} - 2c_4} \right] + \frac{1}{2\sqrt{(1 + \log 2)^2 - 2 \log \bar{x} - 2c_4}} \right\}. \quad (5.11)$$

Here Int stands for the integer closest to the real number enclosed in the braces and we have denoted the integer nearest to the zero of (5.10) by n' . The right side of Eq. (5.11) agrees very well with the numerical results obtained for $\frac{R^*}{x}$: of the 91 points included in Figure 1, the two functions differ by 1 at only one point. The exponential term alone of Eq. (5.11) also agrees very well with the numerical results. We have included the correction, however, to ensure that the expression for the value of the minimum is correct through constant terms.

The asymptotic expansion for $\frac{r^*}{x}$ follows from the asymptotic expansion for $\log h_{n-1}$. The result is

$$\frac{r^*}{x} \sim (\bar{x} e c_3)^n \sqrt{\frac{e}{2n}} n^{n(\log 2 + \frac{1}{2} \log n)} \times \exp \left[O\left(\frac{\log n}{n}\right) \right]. \quad (5.12)$$

It is now straightforward to substitute the right side of Eq. (5.11) into the right side of Eq. (5.12) and obtain an explicit expression for the minimum of $\frac{r^*}{x}$. The result is a lengthy formula which we do not reproduce here. It provides, however, a good approximation to the numerical result. Figure 3 shows the difference between the base-10 logarithm of this analytical formula and the numerical result, as a function of $-\log_{10} \bar{x}$.

Eq. (5.12) is valid for any value of n , not only at the minimum of $\frac{r^*}{x}$. Should we wish to use Eq. (5.10) to simplify (5.12) at the minimum we could, but care should be taken to include the fact that n' is the zero of the right side of (5.10) rounded to the nearest integer.

The difference between n' and the actual zero is $O(\frac{1}{n})$, which in $\frac{r^*}{x}$ gives corrections of order one. Here we then merely note that at the minimum

$$\left. \frac{r^*}{x} \right|_{n=n'} \propto \frac{1}{\sqrt{n'2^{n'}n'^{n'}}} \times \exp \left[O\left(\frac{\log n'}{n'}\right) \right]. \quad (5.13)$$

VI. SUMMARY

In Theorem 4 we have given an upper bound on the norm of vector fields L_n which are computed by requiring that $\mathcal{M}z$ agrees order by order with a given convergent power series. The bound grows with order more rapidly than the exponential function. Thus we cannot use the results of Ref. [5] to ascertain the existence of a finite domain in x for which $\mathcal{M}_n z$ is convergent and analytic as $n \rightarrow \infty$ (analyticity follows from the analytic nesting of domains of successive Lie transformations – see Eq. (3.1) in [5]). Instead, we have sought to optimize the difference between $\mathcal{M}_n z$ and $P(z)$ as a function of n . While an exact analytical expression for the minimum of a bound on the difference proved elusive, at least without significantly weakening the bound or the results of Theorem 4, we have given an asymptotic expression which is valid when the minimum occurs after a large number of Lie transformations. Comparison of asymptotic and numerical results, however, showed close agreement between the two even for values of \bar{x} for which it is optimal to use a relatively small number of Lie transformations. (For example, the results agree to better than 5% for $n_{\min} = 6$.)

It seems well worthwhile to explore now whether the procedure developed in the preceding sections can be adapted to Hamiltonian normal form calculations and used to strengthen the estimates of the type given in Ref. [4] for the norm of generating polynomials. It would also be interesting to examine if the absence of convergence of $\mathcal{M}_n z$ as $n \rightarrow \infty$ is only apparent, due to estimates that were used to obtain a rigorous bound, or is the true property of Lie transformations computed from a power series. The first step in this direction may be the numerical computation of \mathcal{M}_n through a large value of n for some representative $P(z)$.

Note that the calculation of coefficients $a_{rk}^{(n)}$, using Eq. (2.11) or an equivalent, requires only algebraic manipulations, as all derivatives act on powers of z .

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APPENDIX: PROOF OF LEMMA 3.1

We first prove the relation (3.6) by induction. For $s = 0$, $m(l, 0) = l$, $c_{tk}(F, 0) = f_{tk}^{(l)}$, and $B(l, 0) = \phi_l$. Therefore (3.6) reduces to inequality (3.4). For $s = 1$ we have

$$\begin{aligned} L_n F_k^{(l)}(z) &= \sum_{j=1}^d g_j^{(n)} \frac{\partial}{\partial z_j} \sum_{|i|=l} f_{ik}^{(l)} z^i \\ &= \sum_{j=1}^d \sum_{|r|=n} \sum_{|i|=l} a_{rj}^{(n)} f_{ik}^{(l)} i_j z_1^{i_1+r_1} \dots z_j^{i_j+r_j-1} \dots z_d^{i_d+r_d}, \end{aligned} \quad (\text{A1})$$

which yields

$$\begin{aligned} \max_k \left(\sum_{|t|=m(l,1)} |c_{tk}(F, 1)| \right) &\leq \max_k \left(\sum_{j=1}^d \sum_{|r|=n} \sum_{|i|=l} |a_{rj}^{(n)}| |f_{ik}^{(l)}| i_j \right) \\ &\leq \alpha_n \max_k \left(\sum_{j=1}^d \sum_{|i|=l} |f_{ik}^{(l)}| i_j \right) \\ &= \alpha_n l \max_k \left(\sum_{|i|=l} |f_{ik}^{(l)}| \right) \\ &\leq \alpha_n l \phi_l \\ &= B(l, 1), \end{aligned} \quad (\text{A2})$$

as needed. Next, assume that (3.6) holds for a fixed s . Then

$$\begin{aligned} L_n^{s+1} F_k^{(l)}(z) &= \sum_{j=1}^d g_j^{(n)} \frac{\partial}{\partial z_j} \sum_{|t|=m(l,s)} c_{tk}(F, s) z^t \\ &= \sum_{j=1}^d \sum_{|r|=n} \sum_{|t|=m(l,s)} a_{rj}^{(n)} c_{tk}(F, s) t_j z_1^{t_1+r_1} \dots z_j^{t_j+r_j-1} \dots z_d^{t_d+r_d}, \end{aligned} \quad (\text{A3})$$

and we have

$$\begin{aligned}
\max_k \left(\sum_{|t|=m(l,s+1)} |c_{tk}(F, s+1)| \right) &\leq \max_k \left(\sum_{j=1}^d \sum_{|r|=n} \sum_{|t|=m(l,s)} |a_{rj}^{(n)}| |c_{tk}(F, s)| |t_j| \right) \\
&\leq \alpha_n \max_k \left(\sum_{j=1}^d \sum_{|t|=m(l,s)} |c_{tk}(F, s)| |t_j| \right) \\
&= \alpha_n m(l, s) \max_k \left(\sum_{|t|=m(l,s)} |c_{tk}(F, s)| \right) \\
&\leq \alpha_n m(l, s) B(l, s) \\
&= B(l, s+1).
\end{aligned} \tag{A4}$$

The last inequality uses the induction assumption, whereas the last equality follows from the recursion relation satisfied by B . This completes the proof of relation (3.6).

It is now straightforward to establish (3.7). We proceed again by induction. For $s = 0$ we use inequality (3.4) and the special values of m, c , and B given at the beginning of the Appendix to see that (3.7) holds. For the case $s = 1$, on the other hand, we use relations (A1) and (A2) (second inequality) to get

$$\begin{aligned}
\|L_n F_k^{(l)}(z)\| &= \left\| \sum_{j=1}^d \sum_{|r|=n} \sum_{|i|=l} a_{rj}^{(n)} f_{ik}^{(l)} i_j z_1^{i_1+r_1} \dots z_j^{i_j+r_j-1} \dots z_d^{i_d+r_d} \right\| \\
&\leq \max_k \left(\sum_{j=1}^d \sum_{|r|=n} \sum_{|i|=l} |a_{rj}^{(n)}| |f_{ik}^{(l)}| |i_j| \right) x^{n+l-1} \\
&\leq B(l, 1) x^{m(l,1)}.
\end{aligned} \tag{A5}$$

The first inequality is evident from the definition of the norm and for the last relation we have relied on the fact that $m(l, 1) = n + l - 1$. Assume now that (3.7) holds for a fixed s .

With the help of relations (A3) and (A4) (second inequality) we get

$$\begin{aligned}
\|L_n^{s+1} F_k^{(l)}(z)\| &= \left\| \sum_{j=1}^d \sum_{|r|=n} \sum_{|t|=m(l,s)} a_{rj}^{(n)} c_{tk}(F, s) t_j z_1^{t_1+r_1} \dots z_j^{t_j+r_j-1} \dots z_d^{t_d+r_d} \right\| \\
&\leq \max_k \left(\sum_{j=1}^d \sum_{|r|=n} \sum_{|t|=m(l,s)} |a_{rj}^{(n)}| |c_{tk}(F, s)| |t_j| \right) x^{m(l,s)+n-1} \\
&\leq B(l, s+1) x^{m(l,s+1)}.
\end{aligned} \tag{A6}$$

We have again made use of the recursion relation for m , $m(l, s+1) = m(l, s) + n - 1$. This completes the proof of inequality (3.7).

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Figure Captions

Figure 1. n_{\min} vs. $-\log_{10} \bar{x}$. The step size in $\log_{10} \bar{x}$ is 0.1.

Figure 2. The value of $\log_{10} R^*/x$ at $n = n_{\min}$ vs. $-\log_{10} \bar{x}$. The step size in $\log_{10} \bar{x}$ is 0.1, as in Figure 1.

Figure 3. The quantity $\Delta(\bar{x}) = \log_{10} R^*/x|_{n=n_{\min}} - \log_{10} r_a^*/x|_{n=n'_a}$ vs. $-\log_{10} \bar{x}$. Here r_a^*/x denotes the right side of Eq. (5.12) and n'_a the right side of Eq. (5.11). The step size in $\log_{10} \bar{x}$ is 0.1, as in Figures 1 and 2.

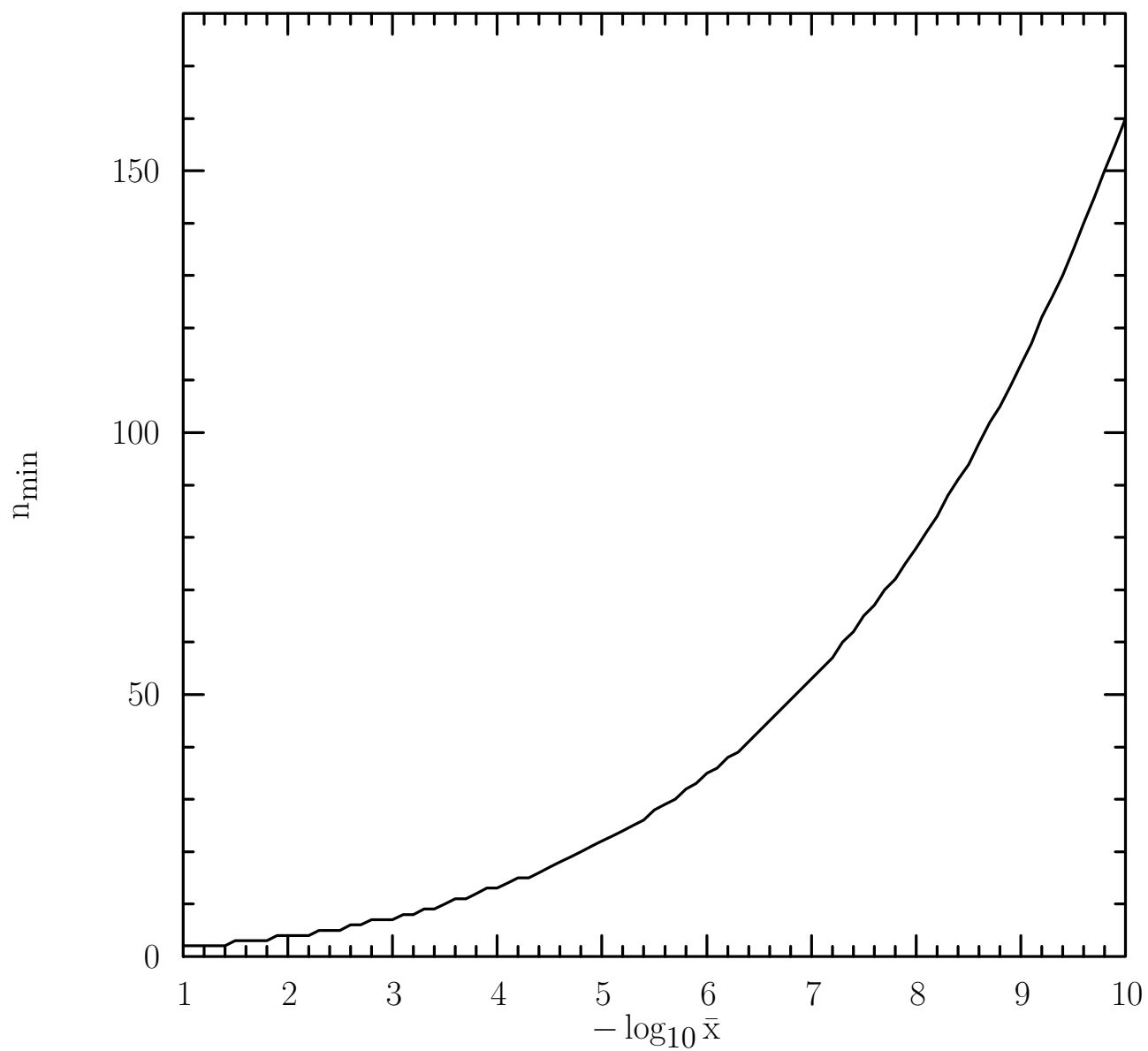


Figure 1

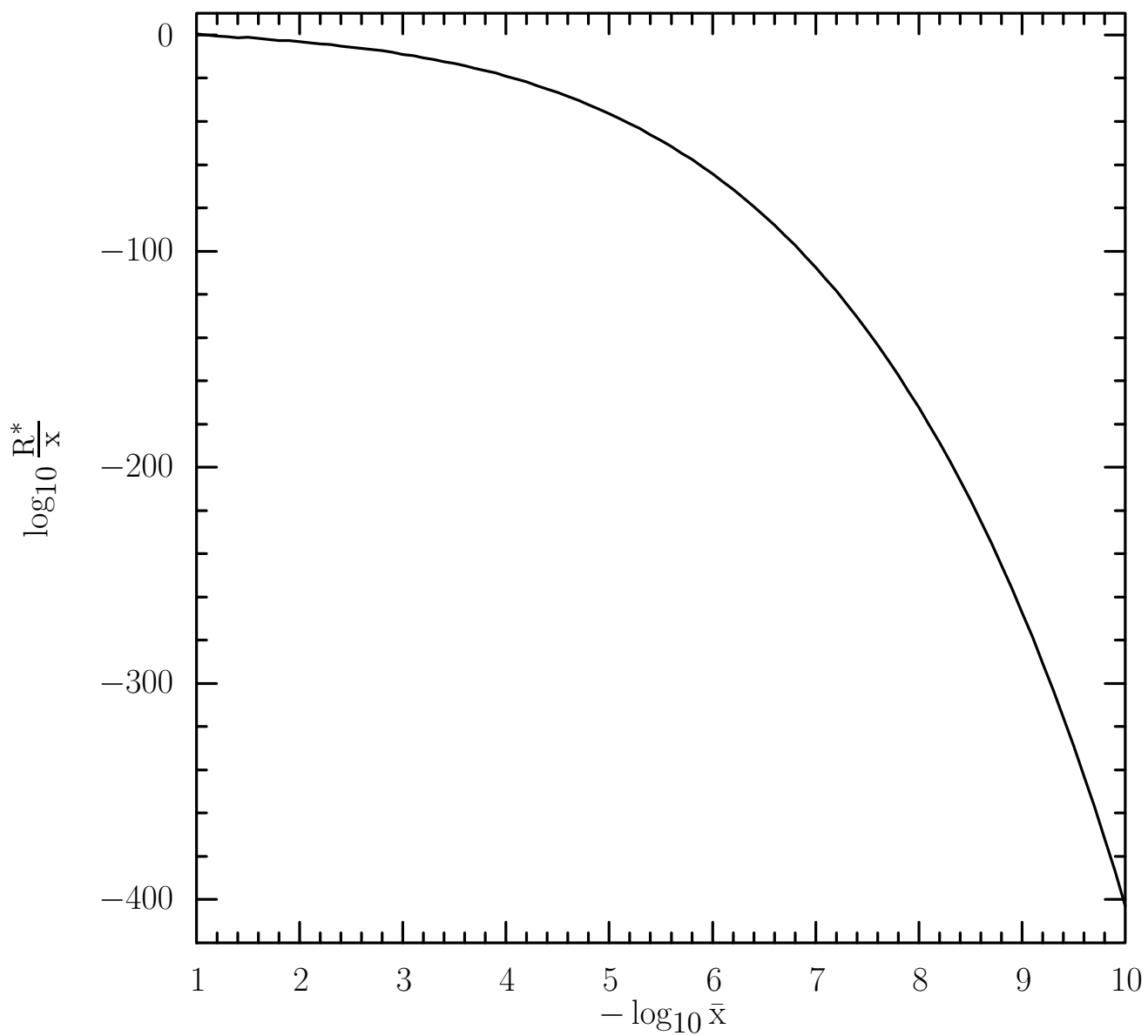


Figure 2

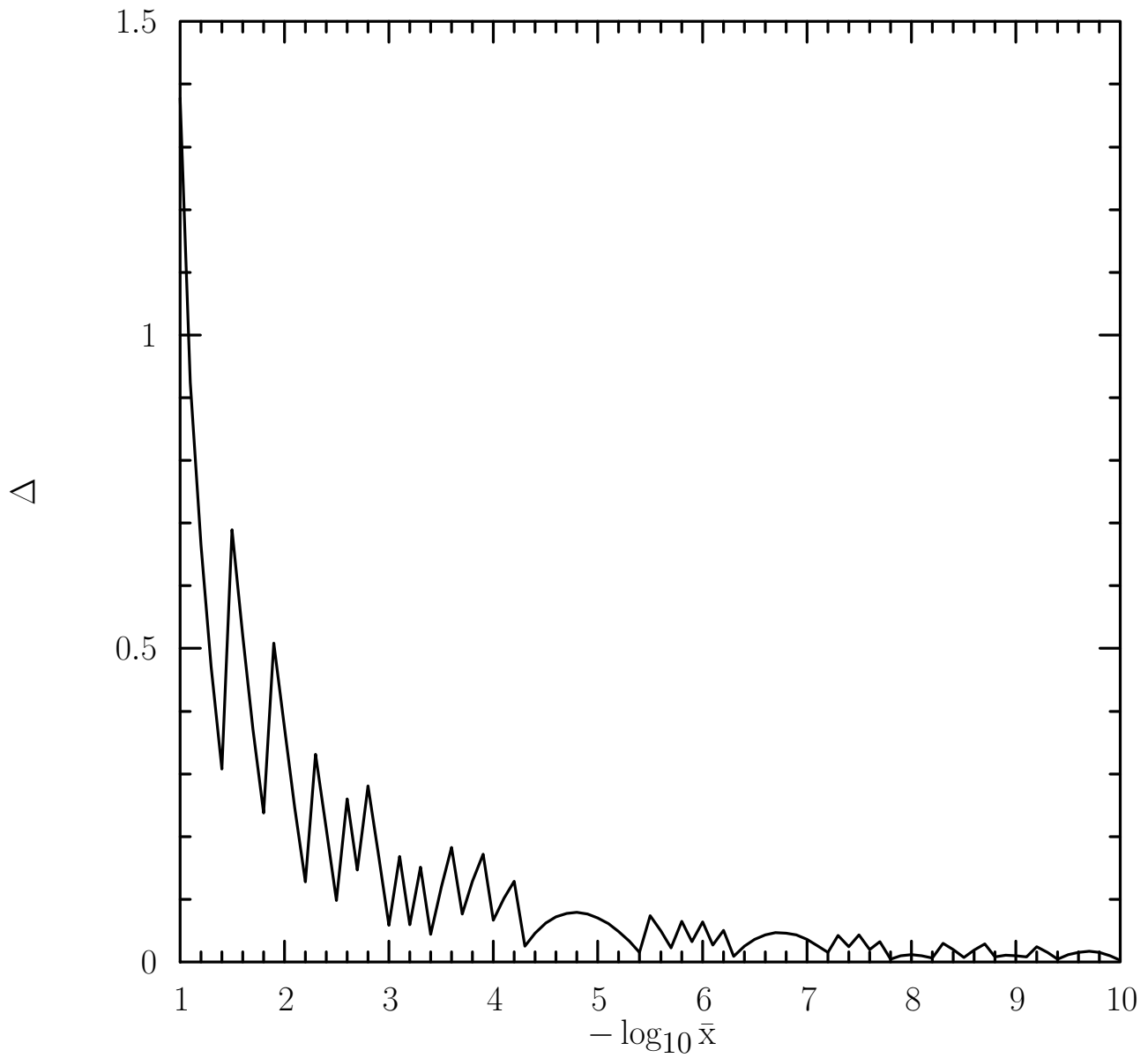


Figure 3